1

Non-Archimedean analytic functions, measures and distributions

In this chapter we give an exposition of some standard facts from the theory of continuous and analytic functions over a non-Archimedean local field. We start by recalling the definitions and notations concerning p-adic and S-adic numbers. Then we discuss the theory of continuous p-adic functions and their p-adic interpolation, and also the basic properties of p-adic analytic functions. In the section 1.3 we introduce distributions and measures, and give a general criterion for the existence of a non-Archimedean measure with given values of integrals of functions belonging to certain dense family ("generalized Kummer congruences"). The next section is devoted to a description of the algebra of bounded measures in terms of their non-Archimedean Mellin transforms (Iwasawa isomorphism). The chapter is completed with an exposition of a general construction of measures, attached to rather arbitrary Euler products. This construction provides a generalization of measures first introduced by Yu.I. Manin (see [Man2]), B. Mazur and H.P.F. Swinnerton-Dyer (see [Maz-SD]). Our construction (see [Pa2] and [Pa3]) was already successfully used in several problems concerning the p-adic analytic interpolation of special values of Dirichlet series (see [Ar], [Co-Schm], [Co-Schn] and [Sc]).

1.1 p-adic numbers and the Tate field

1.1.1 p-adic numbers

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers, (i.e. the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic metric), given by the p-adic valuation

$$\begin{aligned} |\cdot|_p : \ \mathbb{Q} &\longrightarrow \mathbb{R}^+ \\ a/b &\longmapsto |a/b|_p = p^{\mathrm{ord}_p(b)-\mathrm{ord}_p(a)} \\ 0 &\longmapsto |0|_p = 0 \end{aligned}$$

M. Courtieu and A. Panchishkin: LNM 1471, pp. 13–44, 2004.

where $\mathrm{ord}_p(a)$ is the highest power of p dividing the integer a. The function $|\cdot|_p$ is multiplicative, since

$$\mathrm{ord}_p(xy) = \mathrm{ord}_p(x) + \mathrm{ord}_p(y) \tag{1.1}$$

and satisfies the non-Archimedean property

$$|x+y|_p \le \max\left(|x|_p, |y|_p\right). \tag{1.2}$$

If K is a finite algebraic extension of $\mathbb{Q}_p$, then K is generated over $\mathbb{Q}_p$ by a primitive element $\alpha \in K$, so that α is a root of an irreducible polynomial of degree $d = [K : \mathbb{Q}_p]$,

$$f(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Q}_p[x].$$

The valuation $|\cdot|_p$ admits a unique extension to K defined by

$$|\beta|_p = \left(|\mathcal{N}_{K/\mathbb{Q}_p}(\beta)|_p\right)^{1/d}, \tag{1.3}$$

where $\mathcal{N}_{K/\mathbb{Q}_p}(\beta) \in \mathbb{Q}_p$ is the algebraic norm of an element $\beta \in K$. The formula (1.3) defines a unique extension of $|\cdot|_p$ to the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, which satisfies (1.2) (see [Kob2]). The function ord_p is then also extended to $\overline{\mathbb{Q}}_p$ by $\mathrm{ord}_p(\alpha) = \log_p(|\alpha|_p)$. The formula (1.3) implies that $\mathrm{ord}_p(K^\times)$ is an additive subgroup in $\frac{1}{d}\mathbb{Z}$, hence $\mathrm{ord}_p(K^\times) = \frac{1}{e}\mathbb{Z}$ for a certain positive integer e dividing d which is called the *ramification index* of the extension $K/\mathbb{Q}_p$.

Put

$$\mathcal{O}_K = \{x \in K \quad | \quad |x|_p \le 1\}, \qquad M_K = \{x \in K \quad | \quad |x|_p < 1\}. \tag{1.4}$$

Then M_K is the maximal ideal in $\mathcal{O}_K$ and the residue field $\mathcal{O}_K/M_K$ is a finite extension of degree f of $\mathbb{F}_p$ and there is the relation $d = ef$, in which f is called the *inertial degree* of the extension. For each $x \in \mathcal{O}_K$, its *Teichmüller representative* is defined by

$$\omega(x) = \lim_{n\to\infty} x^{p^{fn}}, \qquad \omega(x) \equiv x \pmod{M_K}, \tag{1.5}$$

and satisfies the equation $\omega(x)^{p^f} = \omega(x)$. The map ω provides a homomorphism of the group of invertible elements

$$\mathcal{O}_K^\times = \{x \in K \quad | \quad |x|_p = 1\}$$

of $\mathcal{O}_K$ onto the group of roots of unity of degree $p^f - 1$ in K, denoted by μ_{p^f-1}, and the isomorphism

$$(\mathcal{O}_K/M_K)^\times \xrightarrow{\sim} \mu_{p^f-1} \subset \mathcal{O}_K^\times. \tag{1.6}$$

For example, if $e = 1$ then the extension K is called *unramified*. In this case $f = d$ and the Teichmüller representatives generate K over $\mathbb{Q}_p$, therefore

$K = \mathbb{Q}_p(1^{1/N})$ for $N = p^d - 1$. On the other hand, if $e = d$ then the extension K is called *totally ramified*. For example, if ζ is a primitive root of unity of degree p^n, then $\mathbb{Q}_p(\zeta)$ is totally ramified of degree $d = p^n - p^{n-1}$, and we have that

$$\operatorname{ord}_p(\zeta - 1) = \frac{1}{p^n - p^{n-1}}. \tag{1.7}$$

1.1.2 Topology of p-adic fields

For a field K with the non-Archimedean valuation $|\cdot|_p$ let us define

$$\mathcal{D}_a(r) = \mathcal{D}_a(r;K) = \{x \in K \mid |x-a|_p \le r\}, \tag{1.8}$$

$$\mathcal{D}_a(r^-) = \mathcal{D}_a(r^-;K) = \{x \in K \mid |x-a|_p < r\}, \tag{1.9}$$

("closed" and "open" discs of radius r with center at the point $a \in K$, $r \ge 0$). Then we have that $\mathcal{D}_b(r) = \mathcal{D}_a(r)$ for any $b \in \mathcal{D}_a(r)$ and $\mathcal{D}_b(r^-) = \mathcal{D}_a(r^-)$ for any $b \in \mathcal{D}_a(r^-)$. Note that in the topological sense both discs (1.8) and (1.9) are open and closed subsets of the topological field K.

The important property of the field $\mathbb{Q}_p$ and of its finite extensions is that they are locally compact, so that the discs (1.8) and (1.9) are compact. The disc $\mathbb{Z}_p = \mathcal{D}_0(1, \mathbb{Q}_p)$ is a compact topological ring (the ring of p-adic integers), which is isomorphic to the projective limit of residue rings $\mathbb{Z}/p^n\mathbb{Z}$ with respect to the homomorphisms of reduction modulo p^n:

$$\mathbb{Z}_p = \varprojlim_n (\mathbb{Z}/p^n\mathbb{Z}).$$

Analogously, we have that

$$\mathcal{O}_K = \varprojlim_n (\mathcal{O}_K/M_K^n),$$

and

$$\mathbb{Z}_p^\times = \varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})^\times, \qquad \mathcal{O}_K^\times = \varprojlim_n (\mathcal{O}_K/M_K^n)^\times.$$

1.1.3 The structure of the multiplicative group $\mathbb{Q}_p^\times$ and $K^\times$

Put $\nu = 1$ for $p > 2$ and $\nu = 2$ for $p = 2$, and define

$$U = U_p = \{x \in \mathbb{Z}_p \mid x \equiv 1 \pmod{p^\nu}\}. \tag{1.10}$$

Then there is an isomorphism $U \xrightarrow{\sim} \mathbb{Z}_p$ of the multiplicative group U_p and of the additive group $\mathbb{Z}_p$ which is provided by combining the natural homomorphism

$$U \xrightarrow{\sim} \varprojlim_n \left(U/U^{p^n}\right),$$

and by special isomorphisms $\alpha_{p^n} : U/U^{p^n} \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$, given by

$$\alpha_{p^n}((1+p^\nu)^a) = a \bmod p^n \qquad (a \in \mathbb{Z}). \tag{1.11}$$

One easily verifies that (1.11) is well defined and gives the desired isomorphism. Therefore, the group U is a topological cyclic group, and $(1+p^\nu)$ can be taken as its generator. There are the following decompositions

$$\mathbb{Q}_p = p^{\mathbb{Z}} \times \mathbb{Z}_p^\times, \qquad \mathbb{Z}_p^\times \cong (\mathbb{Z}/p^\nu\mathbb{Z})^\times \times U. \tag{1.12}$$

Similarly, if $[K : \mathbb{Q}_p] = d$, then

$$K^\times = \pi^{\mathbb{Z}} \times \mathcal{O}_K^\times, \qquad \mathcal{O}_K^\times \cong (\mathcal{O}_K/M_K)^\times \times U_K,$$

where π is a generator of the principal ideal $M_K = \pi\mathcal{O}_K$ (i.e. any element $\pi \in K^\times$ with $\mathrm{ord}_p(\pi) = 1/e$),

$$U_K = \{x \in \mathcal{O}_K^\times \quad | \quad |x-1|_p < 1\} = \mathcal{D}_1(1^-; K).$$

The structure of the group U_K is then described as a direct product of d copies of the additive group $\mathbb{Z}_p$ and a finite group consisting of all p-primary roots of unity contained in K (see [Ca-Fr], [Kob2]).

1.1.4 The *S*-adic numbers

Let S be a finite set of prime numbers. For a positive integer M, denote by $S(M)$ its support (i.e. the set of all prime numbers dividing M). Let us consider the projective limit

$$\mathbb{Z}_S = \varprojlim_{M | S(M) \subset S} \mathbb{Z}/M\mathbb{Z},$$

which is taken over all positive integers M with the condition $S(M) \subset S$ with respect to the homomorphisms of reduction. From the Chinese Remainder Theorem (see [Se1], [Weil]) it follows then that

$$\mathbb{Z}_S \cong \prod_{q \in S} \mathbb{Z}_q.$$

Put:

$$\mathbb{Q}_S = \prod_{q \in S} \mathbb{Q}_q, \qquad M_0 = \prod_{q \in S} q.$$

1.1.5 The Tate field

For the purposes of analysis it is convenient to embed $\mathbb{Q}_p$ into a bigger field, which is already complete both in the topological and in the algebraic sence. This field is constructed as the completion $\mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p$ of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ with respect to its single valuation satisfying the condition

$|p|_p = 1/p$. The proof of the fact that $\mathbb{C}_p$ is algebraically closed is not difficult and is based on the Krasner's lemma (see [Kob1],[Kob2], [Wa]).

We shall use the notations

$$\mathcal{O}_p = \{x \in \mathbb{C}_p \mid |x|_p \le 1\}, \quad M_p = \{x \in \mathbb{C}_p \mid |x|_p < 1\} \tag{1.13}$$

Note that $\mathcal{O}_p$ and M_p are no longer compact, and therefore the field $\mathbb{C}_p$ is not locally compact. We also have that

$$\mathcal{O}_p/M_p = \overline{\mathbb{F}}_p \tag{1.14}$$

is the algebraic closure of $\mathbb{F}_p$.

1.2 Continuous and analytic functions over a non-Archimedean field

1.2.1 Continuous functions

Let K be a closed subfield of the Tate field $\mathbb{C}_p$. For a subset $W \subset K$ we consider continuous functions $f : W \to \mathbb{C}_p$. The standard examples of continuous functions are provided by polynomials, by rational functions (at points where they are finite), and also by locally constant functions. If W is compact then for any continuous function $f : W \to \mathbb{C}_p$ and for any $\varepsilon > 0$ there exists a polynomial $h(x) \in \mathbb{C}_p[x]$ such that $|f(x) - h(x)|_p < \varepsilon$ for all $x \in W$. If $f(W) \subset L$ for a closed subfield L of $\mathbb{C}_p$ then $h(x)$ can be chosen so that $h(x) \in L[x]$ (see [Kob1], [Wa]).

Interesting examples of continuous p-adic functions are provided by interpolation of functions, defined on certain subsets, such as $W = \mathbb{Z}$ or $\mathbb{N}$ with $K = \mathbb{Q}_p$. Let f be any function on non-negative integers with values in $\mathbb{Q}_p$ or in some (complete) $\mathbb{Q}_p$-Banach space. In order to extend $f(x)$ to all $x \in \mathbb{Z}_p$ we can use the interpolation polynomials

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}.$$

Then we have that $\binom{x}{n}$ is a polynomial of degree n of x, which for $x \in \mathbb{Z}$, $x \ge 0$ gives the binomial coefficient. If $x \in \mathbb{Z}_p$ then x is close (in the p-adic topology) to a positive integer, hence the value of $\binom{x}{n}$ is also close to an integer, therefore $\binom{x}{n} \in \mathbb{Z}_p$.

The classical Mahler's interpolation theorem says that any continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ can be written in the form (see [La2], [Wa]):

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}, \tag{1.15}$$

with $a_n \to 0$ (p-adically) for $n \to \infty$. For a function $f(x)$ defined for $x \in \mathbb{Z}$, $x \geq 0$ one can write formally

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n},$$

where the coefficients can be founded from the system of linear equations

$$f(n) = \sum_{m=0}^{n} a_m \binom{n}{m},$$

that is

$$a_m = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(j).$$

The series for $f(x)$ is always reduced to a finite sum for each $x \in \mathbb{Z}$, $x \geq 0$. If $a_n \to 0$ then this series is convergent for all $x \in \mathbb{Z}_p$. As was noticed above, the inverse statement is also valid ("*Mahler's criterion*"). If convergence of a_n to zero is so fast that the series defining the coefficients of the x-expansion of $f(x)$ also converge, then $f(x)$ can be extended to an *analytic function*, see 1.2.2 below. Unfortunately, for an arbitrary sequence a_n with $a_n \to 0$ the attempt to use (1.15) for continuation of $f(x)$ out of the subset $\mathbb{Z}_p$ in $\mathbb{C}_p$ may fail. However, in the sequel we mostly consider anlytic functions, that are defined as sums of power series.

1.2.2 Analytic functions and power series

(see [Kob1], p 13). The well known criterion of convergence of a series $\sum_{n=0}^{\infty} a_n$ is that the following partial sums $\sum_{N \leq n \leq M} a_n$ are small for large N, M with $M > N$. In view of the non-Archimedean property (1.2) in $\mathbb{C}_p$ this occurs if and only if $|a_n|_p \to 0$ or $\mathrm{ord}_p(a_n) \to \infty$ for $n \to \infty$. Therefore the convergence of the power series $\sum_{n \geq 0} a_n x^n$ depends only on $|x|_p$ but not on the precise value of x, hence there is no "conditional convergence" in this case. Thus, for any power series $\sum_{n \geq 0} a_n x^n$ we can define its radius of convergence r such that only one of the following holds:

$$\sum_{n=0}^{\infty} a_n x^n \text{ converges} \iff \quad x \in \mathcal{D}_0(r^-), \tag{1.16}$$

$$\sum_{n=0}^{\infty} a_n x^n \text{ converges} \iff \quad x \in \mathcal{D}_0(r). \tag{1.17}$$

An example of the first alternative is $\sum_{n \geq 0} x^n$, where (1.16) is satisfied with $r = 1$, and an example for the second is $\sum_{n \geq 0} p^n x^{p^n - 1}$, where (1.17) is satisfied also with $r = 1$.

The important examples of analytic functions are $\exp(x)$ and $\log(x)$, which are given as the power series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n, \tag{1.18}$$

and we have that

$$\exp(x) \text{ converges on } \mathcal{D}_0(p^{-1/(p-1)-}), \qquad r = p^{-1/(p-1)}, \tag{1.19}$$

and

$$\log(1+x) \text{ converges on } \mathcal{D}_0(1^-), \qquad r = 1, \tag{1.20}$$

(see [Kob1], [Bo-Ša]), so that $\exp(x)$ converges in a disc smaller than the unit disc, and $\log(1+x)$ has better convergence that $\exp(x)$.

Since the identity

$$\log(xy) = \log(x) + \log(y) \tag{1.21}$$

holds as a formal power series identity, it follows that (1.21) holds in $\mathbb{C}_p$ as long as $|x-1|_p < 1$ and $|y-1|_p < 1$. In particular, since $|\zeta - 1|_p < 1$ for ζ any p^n-root of unity, we can obviously apply (1.21) to conclude that $\log(\zeta) = 0$. Also we have that for all $x \in \mathcal{D}_0(p^{-1/(p-1)-})$ the following identities hold

$$\exp(\log(1+x)) = 1 + x, \text{ and } \log(\exp(x)) = x,$$

which are deduced from the corresponding properties of the formal series and can be used for establishing isomorphisms between certain additive and multiplicative subgroups in $\mathbb{C}_p$ and $\mathbb{C}_p^\times$; for $U = \mathcal{D}_1(p^{\nu-}; \mathbb{Q}_p)$ (with ν as in 1.1.3) there are the isomorphisms

$$\exp : p^{\nu+n}\mathbb{Z}_p \xrightarrow{\sim} U^{p^n} \qquad (\text{with } n \geq 0). \tag{1.22}$$

Theorem 1.1 (On analyticity of interpolation series). *Let $r < p^{-1/(p-1)} < 1$, and let*

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n},$$

be a series with the condition $|a_n|_p \leq M r^n$ for some $M > 0$. Then $f(x)$ is expresible as a certain power series whose radius of convergence is not less than $R = (rp^{1/(p-1)})^{-1} > 1$.

Proof. (see [Wa], p. 53)

As an example let us consider the function $\langle a \rangle^x$ which is defined for $a \in \mathbb{Z}_p^\times$ by means of the decomposition $a = \omega(a)\langle a \rangle$ where $\omega(a)$ is the Teichmüller representative of a for $p > 2$ and $\omega(a) = \pm 1$ for $p = 2$ with $\omega(a) \equiv a \pmod 4$, and the exponentiation is given by the binomial formula

$$\langle a\rangle^x = (1+\langle a\rangle - 1)^x = \sum_{n=0}^{\infty} \binom{x}{n} (\langle a\rangle - 1)^n. \tag{1.23}$$

Since $|\langle a\rangle - 1|_p \le p^{\nu}$ we may put in the above theorem $r = p^{-\nu}$ and get that the function $\langle a\rangle^x$ is a power series in x with the radius of convergence not less than $p^{\nu-1/(p-1)} > 1$ and the following equality holds:

$$\exp(x\log(\langle a\rangle)) = \sum_{n=0}^{\infty} \binom{x}{n} (\langle a\rangle - 1)^n. \tag{1.24}$$

in which both parts are analytic in x and coincide for $x \in \mathbb{N}$.

1.2.3 Newton polygons

(see [Kob1], p 21). The *Newton polygon* M_f for a power series

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \in \mathbb{C}_p[[x]]$$

is defined as the convex hull of the points $(n, \mathrm{ord}_p(a_n))$ (where we agree to take $\mathrm{ord}_p(0) = \infty$).

It is not hard to prove the following.

Proposition 1.2. *If a segment of M_f has slope λ and horizontal length N (i.e. it extends from $(n, \mathrm{ord}_p(a_n))$ to $(n+N, \lambda N + \mathrm{ord}_p(a_n))$ then f has precisely N roots r_n with $\mathrm{ord}_p(r_n) = -\lambda$ (counting multiplicity).*

The following theorem is the p-adic analog of the Weierstrass Preparation theorem (see [Kob1], p. 21).

Theorem 1.3. *Let*

$$f(x) = a_m x^m + a_{m+1}x^{m+1} + \cdots \in \mathbb{C}_p[[x]], \quad a_m \ne 0$$

be a power series which converges on $\mathcal{D}_0(p^{\lambda}; \mathbb{C}_p)$. Let $(N, \mathrm{ord}_p(a_N))$ be the right endpoint of the last segment of M_f with slope $\le \lambda$, if this N is finite. Otherwise, there will be a last infinitely long segment of slope λ and only finitely many points $(n, \mathrm{ord}_p(a_n))$ on that segment. In that case let N be the last such n. Then there exists a unique polynomial $h(x)$ of the form

$$b_m x^m + b_{m+1}x^{m+1} + \cdots + b_N x^N$$

with $b_m = a_m$ and a unique power series $g(x)$ which converges and does not vanish on $\mathcal{D}_0(p^{\lambda}; \mathbb{C}_p)$ such that

$$f(x) = \frac{h(x)}{g(x)} \text{ on } \mathcal{D}_0(p^{\lambda}; \mathbb{C}_p).$$

In addition M_h coincide with M_f as far as the point $(N, \mathrm{ord}_p(a_N))$.

Corollary 1.4. *A power series which converges everywhere and has no zeroes is a constant.*

A simple proof of the Weierstrass Preparation Theorem for power series of the type

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \in \mathcal{O}_p[[x]]$$

is based on a generalization of the Euclid algorithm (see [Man1]).

There exists another definition (dual) of the Newton polygon (see [Kob1], [Vi1]) of a series

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \in \mathbb{C}_p[[x]].$$

Instead of the points $(n, \mathrm{ord}_p(a_n))$ let us look at the lines $l_n : y = nx + \mathrm{ord}_p(a_n)$ with slope n. Then $\widetilde{M_f}$ is defined as the graph of the function $\min_n l_n(x)$. The x-coordinate of the points of intersection of the l_n which appear in $\widetilde{M_f}$ give ord_p of the zeroes, and the difference between the slopes n of the successive l_n which appear in $\widetilde{M_f}$ give the number of zeroes with given ord_p. This definition is explained by the fact that $\widetilde{M_f}$ coincides with the graph of the function

$$\widetilde{M_f}(t) = \log_p \left(\sup_{|x|_p < r} |f(x)|_p \right)$$

which is piecewise linear in the variable $t = \log_p(r)$. This function is given on the intervals of linearity by the formula $\log_p |a_i|_p + it$ where $a_i x^i$ is the leading term (i.e. maximum in module) in the expansion of f.

The first obvious application of this other type of Newton polygon is that a function

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \in \mathbb{C}_p[[x]]$$

is bounded in the open disc $\mathcal{D}_0(1^-; \mathbb{C}_p)$ if and only if the coefficents a_n are uniformly bounded.

1.3 Distributions, measures, and the abstract Kummer congruences

1.3.1 Distributions

Let us consider a commutative associative ring R, an R-module $\mathcal{A}$ and a profinite (i.e. compact and totally disconnected) topological space Y. Then Y is a projective limit of finite sets:

$$Y = \varprojlim_{I} Y_i$$

where I is a (partially ordered) inductive set and for $i \geq j$, i, $j \in I$ there are surjective homomorphisms $\pi_{i,j} : Y_i \to Y_j$ with the condition $\pi_{i,j} \circ \pi_{j,k} = \pi_{i,k}$ for $i \geq j \geq k$. The inductivity of I means that for any $i, j \in I$ there exists $k \in I$ with the condition $k \geq i$, $k \geq j$. By the universal property we have that for each $i \in I$ a unique map $\pi_i : Y \to Y_i$ is defined, which satisfies the property $\pi_{i,j} \circ \pi_i = \pi_j$ (for each $i, j \in I$).

Let $\mathrm{Step}(Y, R)$ be the R-module consisting of all R-valued locally constant functions $\phi : Y \to R$.

Definition 1.5. *A distribution on Y with values in a R-module $\mathcal{A}$ is a R-linear homomorphism*

$$\mu : \mathrm{Step}(Y, R) \longrightarrow \mathcal{A}.$$

For $\varphi \in \mathrm{Step}(Y, R)$ we use the notations

$$\mu(\varphi) = \int_Y \varphi \mathrm{d}\mu = \int_Y \varphi(y)\mathrm{d}\mu(y).$$

Each distribution μ can be defined by a system of functions $\mu^{(i)} : Y_i \to \mathcal{A}$, satisfying the following finite-additivity condition

$$\mu^{(j)}(y) = \sum_{x \in \pi_{i,j}^{-1}(y)} \mu^{(i)}(x) \qquad (y \in Y_j, \quad x \in Y_i). \tag{1.25}$$

In order to construct such a system it suffices to put

$$\mu^{(i)}(x) = \mu(\delta_{i,x}) \in \mathcal{A} \qquad (x \in Y_i),$$

where $\delta_{i,x}$ is the characteritic function of the inverse image $\pi_i^{-1}(x) \subset Y$ with respect to the natural projection $Y \to Y_i$. For an arbitrary function $\varphi_j : Y_j \to R$ and $i \geq j$ we define the functions

$$\varphi_i = \varphi_j \circ \pi_{i,j}, \qquad \varphi = \varphi_j \circ \pi_j, \qquad \varphi \in \mathrm{Step}(Y, R), \qquad \varphi_i : Y_i \xrightarrow{\pi_{i,j}} Y_j \longrightarrow R.$$

A convenient criterion of the fact that a system of functions $\mu^{(i)} : Y_i \to \mathcal{A}$ satisfies the finite additivity condition (1.25) (and hence is associated to some distribution) is given by the following condition (*compatibility criterion*): for all $j \in I$, and $\varphi_j : Y_j \to R$ the value of the sums

$$\mu(\varphi) = \mu^{(i)}(\varphi_i) = \sum_{y \in Y_i} \varphi_i(y)\mu^{(i)}(y), \tag{1.26}$$

is independent of i for all large enough $i \geq j$. When using (1.26), it suffices to verify the condition (1.26) for some “basic” system of functions. For example, if

$$Y = G = \varprojlim_i G_i$$

is a profinite abelian group, and R is a domain containing all roots of unity of the order dividing the order of Y (which is a "supernatural number") then it suffices to check the condition (1.26) for all characters of finite order $\chi : G \to R$, since their $R \otimes \mathbb{Q}$ -linear span coincides with the whole ring $\mathrm{Step}(Y, R \otimes \mathbb{Q})$ by the orthogonality properties for characters of a finite group (see [Kat3], [Maz-SD]).

Example: Bernoulli distributions (see [La1]). Let M be a positive integer, $f : \mathbb{Z} \to \mathbb{C}$ is a periodic function with the period M (i.e. $f(x + M) = f(x)$, $f : \mathbb{Z}/M\mathbb{Z} \to \mathbb{C}$). The generalized Bernoulli number (see [Le1], [Šaf]) $B_{k,f}$ is defined as $k!$ times the coefficient by t^k in the expansion in t of the following rational quotient

$$\sum_{a=0}^{M-1} \frac{f(a)te^{at}}{e^{Mt} - 1},$$

that is,

$$\sum_{k=0}^{\infty} \frac{B_{k,f}}{k!} t^k = \sum_{a=0}^{M-1} \frac{f(a)te^{at}}{e^{Mt} - 1}. \tag{1.27}$$

Now let us consider the profinite ring

$$Y = \mathbb{Z}_S = \varprojlim_M (\mathbb{Z}/M\mathbb{Z})$$

($S(M) \subset S$), the projective limit being taken over the set of all positive integers M with support $S(M)$ in a fixed finite set S of prime numbers. Then the periodic function $f : \mathbb{Z}/M\mathbb{Z} \to \mathbb{C}$ with $S(M) \subset S$ may be viewed as an element of $\mathrm{Step}(Y, \mathbb{C})$. We claim that there exists a distribution E_k : $\mathrm{Step}(Y, \mathbb{C}) \to \mathbb{C}$ which is uniquely determined by the condition

$$E_k(f) = B_{k,f} \text{ for all } f \in \mathrm{Step}(Y, \mathbb{C}). \tag{1.28}$$

In order to prove the existence of this distribution we use the above criterion (1.26) and check that for every $f \in \mathrm{Step}(Y, \mathbb{C})$ the right hand side in (1.28) (i.e. $B_{k,f}$) does not depend on the choice of a period M of the function f. It follows directly from the definition (1.27); however we give here a different proof which is based on an interpretation of the numbers $B_{k,f}$ as certain special values of L-functions.

For the function $f : \mathbb{Z}/M\mathbb{Z} \to \mathbb{C}$ let

$$L(s, f) = \sum_{n=1}^{\infty} f(n)n^{-s}$$

be the corresponding L-series which is absolutely convergent for all s with $\mathrm{Re}(s) > 1$ and admits an analytic continuation over all $s \in \mathbb{C}$. For this series we have that

$$L(1-k, f) = -\frac{B_{k,f}}{k}. \tag{1.29}$$

For example, if $f \equiv 1$ is the constant function with the period $M = 1$ then we have that

$$\zeta(1-k) = -\frac{B_k}{k}, \quad \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k = \frac{t}{e^t - 1},$$

B_k being the Bernoulli number. The formula (1.29) is established by means of the contour integral discovered by Riemann (see [La1]). This formula apparently implies the desired independence of $B_{k,f}$ on the choice of M. We note also that if $K \subset \mathbb{C}$ is an arbitrary subfield, and $f(Y) \subset K$ then we have from the formula (1.27) that $B_{k,f} \in K$ hence the distribution E_k is a K-valued distribution on Y.

1.3.2 Measures

Let R be a topological ring, and $\mathcal{C}(Y, R)$ be the topological module of all R-valued functions on a profinite set Y.

Definition 1.6. *A measure on Y with values in the topological R-module $\mathcal{A}$ is a continuous homomorphism of R-modules*

$$\mu : \mathcal{C}(Y, R) \longrightarrow \mathcal{A}.$$

The restriction of μ to the R-submodule $\mathrm{Step}(Y, R) \subset \mathcal{C}(Y, R)$ defines a distribution which we denote by the same letter μ, and the measure μ is uniquely determined by the corresponding distribution since the R-submodule $\mathrm{Step}(Y, R)$ is dense in $\mathcal{C}(Y, R)$. The last statement expresses the well known fact about the uniform continuity of a continuous function over a compact topological space.

Now we consider any closed subring R of the Tate field $\mathbb{C}_p$, $R \subset \mathbb{C}_p$, and let $\mathcal{A}$ be a complete R-module with topology given by a norm $|\cdot|_{\mathcal{A}}$ on $\mathcal{A}$ compatible with the norm $|\cdot|_p$ on $\mathbb{C}_p$ so that the following conditions are satisfie:

- for $x \in \mathcal{A}$ the equality $|x|_{\mathcal{A}} = 0$ is equivalent to $x = 0$,
- for $a \in R$, $x \in \mathcal{A}$: $|ax|_{\mathcal{A}} = |a|_p |x|_{\mathcal{A}}$,
- for all $x, y \in \mathcal{A}$: $|x + y|_{\mathcal{A}} < \max(|x|_{\mathcal{A}}, |y|_{\mathcal{A}})$.

Then the fact that a distribution (a system of functions $\mu^{(i)} : Y_i \to \mathcal{A}$) gives rise to a $\mathcal{A}$-valued measure on Y is equivalent to the condition that the system $\mu^{(i)}$ is bounded, i.e. for some constant $B > 0$ and for all $i \in I$, $x \in Y_i$ the following uniform estimate holds:

$$|\mu^{(i)}(x)|_{\mathcal{A}} < B. \tag{1.30}$$

This criterion is an easy consequence of the non-Archimedean property

$$|x + y|_{\mathcal{A}} \leq \max(|x|_{\mathcal{A}}, |y|_{\mathcal{A}})$$

of the norm $|\cdot|_{\mathcal{A}}$ (see [Man2], [Vi1]).

In particular if $\mathcal{A} = R = \mathcal{O}_p = \{x \in \mathbb{C}_p \quad | \quad |x|_p \leq 1\}$ is the subring of integers in the Tate field $\mathbb{C}_p$ then the set of $\mathcal{O}_p$-valued distributions on Y coincides with $\mathcal{O}_p$-valued measures (in fact, both sets are R-algebras with multiplication defined by convolution, see section 1.4).

Below we give some examples of measures based on the following important criterion of existence of a measure with given properties.

Proposition 1.7 (The abstract Kummer congruences). *(see [Kat3]). Let $\{f_i\}$ be a system of continuous functions $f_i \in \mathcal{C}(Y, \mathcal{O}_p)$ in the ring $\mathcal{C}(Y, \mathcal{O}_p)$ of all continuous functions on the compact totally disconnected group Y with values in the ring of integers $\mathcal{O}_p$ of $\mathbb{C}_p$ such that $\mathbb{C}_p$-linear span of $\{f_i\}$ is dense in $\mathcal{C}(Y, \mathbb{C}_p)$. Let also $\{a_i\}$ be any system of elements $a_i \in \mathcal{O}_p$. Then the existence of an $\mathcal{O}_p$-valued measure μ on Y with the property*

$$\int_Y f_i d\mu = a_i$$

is equivalent to the following congruences, for an arbitrary choice of elements $b_i \in \mathbb{C}_p$ almost all of which vanish

$$\sum_i b_i f_i(y) \in p^n \mathcal{O}_p \text{ for all } y \in Y \text{ implies } \sum_i b_i a_i \in p^n \mathcal{O}_p. \tag{1.31}$$

Remark 1.8. Since $\mathbb{C}_p$-measures are characterised as bounded $\mathbb{C}_p$-valued distributions, every $\mathbb{C}_p$-measures on Y becomes a $\mathcal{O}_p$-valued measure after multiplication by some non-zero constant.

Proof of proposition 1.7. The nessecity is obvious since

$$\begin{aligned} \sum_i b_i a_i = & \int_Y (p^n \mathcal{O}_p - \text{valued function}) d\mu = \\ & = p^n \int_Y (\mathcal{O}_p - \text{valued function}) d\mu \in p^n \mathcal{O}_p. \end{aligned}$$

In order to prove the sufficiency we need to construct a measure μ from the numbers a_i. For a function $f \in \mathcal{C}_(Y, \mathcal{O}_p)$ and a positive integer n there exist elements $b_i \in \mathbb{C}_p$ such that only a finite number of b_i does not vanish, and

$$f - \sum_i b_i f_i \in p^n \mathcal{C}(Y, \mathcal{O}_p),$$

according to the density of the $\mathbb{C}_p$-span of $\{f_i\}$ in $\mathcal{C}(Y, \mathbb{C}_p)$. By the assumption (1.31) the value $\sum_i a_i b_i$ belongs to $\mathcal{O}_p$ and is well defined modulo p^n (i.e. does not depend on the choice of b_i). Following N.M. Katz ([Kat3]), we denote this value by "$\int_Y f d\mu \bmod p^n$ ". Then we have that the limit procedure

$$\int_Y f d\mu = \lim_{n\to\infty} \text{“} \int_Y f d\mu \bmod p^n \text{”} \in \varprojlim_n \mathcal{O}_p / p^n \mathcal{O}_p = \mathcal{O}_p,$$

gives the measure μ.

1.3.3 The S-adic Mazur measure

Let $c > 1$ be a positive integer coprime to

$$M_0 = \prod_{q \in S} q$$

with S being a fixed set of prime numbers. Using the criterion of the proposition 1.7 we show that the $\mathbb{Q}$ -valued distribution defined by the formula

$$E_k^c(f) = E_k(f) - c^k E_k(f_c), \qquad f_c(x) = f(cx), \tag{1.32}$$

turns out to be a measure where $E_k(f)$ are defined in 1.3.1, $f \in \mathrm{Step}(Y, \mathbb{Q}_p)$ and the field $\mathbb{Q}$ is regarded as subfield of $\mathbb{C}_p$. Define the generelized Bernoulli polynomials $B_{k,f}^{(M)}(X)$ as

$$\sum_{k=0}^{\infty} B_{k,f}^{(M)}(X)\frac{t^k}{k!} = \sum_{a=0}^{M-1} f(a)\frac{te^{(a+X)t}}{e^{Mt}-1}, \tag{1.33}$$

and the generalized sums of powers

$$S_{k,f}(M) = \sum_{a=0}^{M-1} f(a)a^k.$$

Then the definition (1.33) formally implies that

$$\frac{1}{k}[B_{k,f}^{(M)}(M) - B_{k,f}^{(M)}(0)] = S_{k-1,f}(M), \tag{1.34}$$

and also we see that

$$B_{k,f}^{(M)}(X) = \sum_{i=0}^{k} \binom{k}{i} B_{i,f}X^{k-i} = B_{k,f} + kB_{k-1,f}X + \cdots + B_{0,f}X^k. \tag{1.35}$$

The last identity can be rewritten symbolically as

$$B_{k,f}(X) = (B_f + X)^k.$$

The equality (1.34) enables us to calculate the (generalized) sums of powers in terms of the (generalized) Bernoulli numbers. In particular this equality implies that the Bernoulli numbers $B_{k,f}$ can be obtained by the following p-adic limit procedure (see [La1]):

$$B_{k,f} = \lim_{n\to\infty} \frac{1}{Mp^n} S_{k,f}(Mp^n) \quad \text{(a p-adic limit)}, \tag{1.36}$$

where f is a $\mathbb{C}_p$-valued function on $Y = \mathbb{Z}_S$. Indeed, if we replace M in (1.34) by Mp^n with growing n and let D be the common denominator of all coefficients of the polynomial $B_{k,f}^{(M)}(X)$. Then we have from (1.35) that

$$\frac{1}{k}\left[B_{k,f}^{(Mp^n)}(M) - B_{k,f}^{(M)}(0)\right] \equiv B_{k-1,f}(Mp^n) \pmod{\frac{1}{kD}p^2 n}. \tag{1.37}$$

The proof of (1.36) is accomplished by division of (1.37) by Mp^n and by application of the formula (1.34).

Now we can directly show that the distribution E_k^c defined by (1.32) are in fact bounded measures. If we use (1.31) and take the functions $\{f_i\}$ to be all of the functions in $\mathrm{Step}(Y, \mathcal{O}_p)$. Let $\{b_i\}$ be a system of elements $b_i \in \mathbb{C}_p$ such that for all $y \in Y$ the congruence

$$\sum_i b_i f_i(y) \equiv 0 \pmod{p^n} \tag{1.38}$$

holds. Set $f = \sum_i b_i f_i$ and assume (without loss of generality) that the number n is large enough so that for all i with $b_i \neq 0$ the congruence

$$B_{k,f_i} \equiv \frac{1}{Mp^n} S_{k,f_i}(Mp^n) \pmod{p^n} \tag{1.39}$$

is valid in accordance with (1.36). Then we see that

$$B_{k,f} \equiv (Mp^n)^{-1} \sum_i \sum_{a=0}^{Mp^n-1} b_i f_i(a) a^k \pmod{p^n}, \tag{1.40}$$

hence we get by definition (1.32):

$$\begin{aligned} E_k^c(f) &= B_{k,f} - c^k B_{k,f_c} \\ &\equiv (Mp^n)^{-1} \sum_i \sum_{a=0}^{Mp^n-1} b_i \left[f_i(a)a^k - f_i(ac)(ac)^k\right] \pmod{p^n}. \end{aligned} \tag{1.41}$$

Let $a_c \in \{0, 1, \cdots, Mp^n - 1\}$, such that $a_c \equiv ac \pmod{Mp^n}$, then the map $a \longmapsto a_c$ is well defined and acts as a permutation of the set $\{0, 1, \cdots, Mp^n - 1\}$, hence (1.41) is equivalent to the congruence

$$E_k^c(f) = B_{k,f} - c^k B_{k,f_c} \equiv \sum_i \frac{a_c^k - (ac)^k}{Mp^n} \sum_{a=0}^{Mp^n-1} b_i f_i(a) a^k \pmod{p^n}. \tag{1.42}$$

Now the assumption (1.37) formally inplies that $E_k^c(f) \equiv 0 \pmod{p^n}$, completing the proof of the abstact congruences and the construction of measures E_k^c.

Remark 1.9. The formula (1.41) also implies that for all $f \in \mathcal{C}(Y, \mathbb{C}_p)$ the following holds

$$E_k^c(f) = k E_1^c(x_p^{k-1} f) \tag{1.43}$$

where $x_p : Y \longrightarrow \mathbb{C}_p \in \mathcal{C}(Y, \mathbb{C}_p)$ is the composition of the projection $Y \longrightarrow \mathbb{Z}_p$ and the embedding $\mathbb{Z}_p \hookrightarrow \mathbb{C}_p$.

Indeed if we put $a_c = ac + Mp^n t$ for some $t \in \mathbb{Z}$ then we see that

$$a_c^k - (ac)^k = (ac + Mp^n t)^k - (ac)^k \equiv kMp^n t(ac)^{k-1} \pmod{(Mp^n)^2},$$

and we get that in (1.42):

$$\frac{a_c^k - (ac)^k}{Mp^n} \equiv k(ac)^{k-1} \frac{a_c - ac}{Mp^n} \pmod{Mp^n}.$$

The last congruence is equivalent to saying that the abstract Kummer congruences (1.31) are satisfied by all functions of the type $x_p^{k-1} f_i$ for the measure E_1^c with $f_i \in \mathrm{Step}(Y, \mathbb{C}_p)$ establishing the identity (1.43).

1.4 Iwasawa algebra and the non-Archimedean Mellin transform

1.4.1 The domain of definition of the non-Archimedean zeta functions

In the classical case the set on which zeta functions are defined is the set of complex numbers $\mathbb{C}$ which may be viewed equally as the set of all continuous characters (more precisely, quasicharacters) via the following isomorphism:

$$\begin{aligned} \mathbb{C} &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}(\mathbb{R}_+^\times, \mathbb{C}^\times) \\ s &\longmapsto (y \longmapsto y^s) \end{aligned} \tag{1.44}$$

The construction which associates to a function $h(y)$ on $\mathbb{R}_+^\times$ (with certain growth conditions as $y \to \infty$ and $y \to 0$) the following integral

$$L_h(s) = \int_{\mathbb{R}_+^\times} h(y) y^s \frac{dy}{y}$$

(which converges probably not for all values of s) is called the *Mellin transform*. For example, if $\zeta(s) = \sum_{n \geq 1} n^{-s}$ is the Riemann zeta function, then the function $\zeta(s)\Gamma(s)$ is the Mellin transform of the function $h(y) = 1/(1 - e^{-y})$:

$$\zeta(s)\Gamma(s) = \sum_0^\infty \frac{1}{1 - e^{-y}} y^s \frac{dy}{y}, \tag{1.45}$$

so that the integral and the series are absolutely convergent for $\mathrm{Re}(s) > 1$. For an arbitrary function of type

$$f(z) = \sum_{n=1}^\infty a(n) e^{2i\pi n z}$$

with $z = x + iy \in \mathbb{H}$ in the upper half plane $\mathbb{H}$ and with the growth condition $a(n) = \mathcal{O}(n^c)$ $(c > 0)$ on its Fourier coefficients, we see that the zeta function

$$L(s, f) = \sum_{n=1}^{\infty} a(n) n^{-s},$$

essentially coincides with the Mellin transform of $f(z)$, that is

$$\frac{\Gamma(s)}{(2\pi)^s} L(s, f) = \int_0^{\infty} f(iy) y^s \frac{dy}{y}. \tag{1.46}$$

Both sides of the equality (1.46) converge absolutely for $\mathrm{Re}(s) > 1 + c$. The identities (1.45) and (1.46) are immediately deduced from the well known integral representation for the gamma-function

$$\Gamma(s) = \int_0^{\infty} e^{-y} y^s \frac{dy}{y}, \tag{1.47}$$

where $\frac{dy}{y}$ is a measure on the group $\mathbb{R}_+^\times$ which is invariant under the group translations (Haar measure). The integral (1.47) is absolutely convergent for $\mathrm{Re}(s) > 0$ and it can be interpreted as the integral of the product of an additive character $y \mapsto e^{-y}$ of the group $\mathbb{R}^{(+)}$ restricted to $\mathbb{R}_+^\times$, and of the multiplicative character $y \mapsto y^s$, integration is taken with respect to the Haar measure $\mathrm{d}y/y$ on the group $\mathbb{R}_+^\times$.

In the theory of the non-Archimedean integration one considers the group $\mathbb{Z}_S^\times$ (the group of units of the S-adic completion of the ring of integers $\mathbb{Z}$) instead of the group $\mathbb{R}_+^\times$, and the Tate field $\mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p$ (the completion of an algebraic closure of $\mathbb{Q}_p$) instead of the complex field $\mathbb{C}$. The domain of definition of the p-adic zeta functions is the p-adic analytic group

$$X_S = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_S^\times, \mathbb{C}_p^\times) = X(\mathbb{Z}_S^\times), \tag{1.48}$$

where:

$$\mathbb{Z}_S^\times \cong \oplus_{q \in S} \mathbb{Z}_q^\times,$$

and the symbol

$$X(G) = \mathrm{Hom}_{\mathrm{cont}}(G, \mathbb{C}_p^\times) \tag{1.49}$$

denotes the functor of all p-adic characters of a topological group G (see [Vi1]).

1.4.2 The analytic structure of X_S

Let us consider in detail the structure of the topological group X_S. Define

$$U_p = \{x \in \mathbb{Z}_p^\times \quad | \quad x \equiv 1 \pmod{p^\nu}\},$$

where $\nu = 1$ or $\nu = 2$ according as $p > 2$ or $p = 2$. Then we have the natural decomposition

$$X_S = X\left((\mathbb{Z}/p^\nu\mathbb{Z})^\times \times \prod_{q\neq p}\mathbb{Z}_q^\times\right) \times X(U_p). \tag{1.50}$$

The analytic dstructure on $X(U_p)$ is defined by the following isomorphism (which is equivalent to a special choice of a local parameter):

$$\varphi : X(U_p) \xrightarrow{\sim} T = \{z \in \mathbb{C}_p^\times \quad | \quad |z-1|_p < 1\},$$

where $\varphi(x) = x(1+p^\nu)$, $1+p^\nu$ being a topoplogical generator of the multiplicative group $U_p \cong \mathbb{Z}_p$. An arbitrary character $\chi \in X_S$ can be uniquely represented in the form $\chi = \chi_0\chi_1$ where χ_0 is trivial on the component U_p, and χ_1 is trivial on the other component

$$(\mathbb{Z}/p^\nu\mathbb{Z})^\times \times \prod_{q\neq p}\mathbb{Z}_q^\times.$$

The character χ_0 is called the *tame component*, and χ_1 the *wild component* of the character χ. We denote by the symbol $\chi_{(t)}$ the (wild) character which is uniquely determined by the condition

$$\chi_{(t)}(1+p^\nu) = t$$

with $t \in \mathbb{C}_p$, $|t|_p < 1$.

In some cases it is convenient to use another local coordinate s which is analogous to the classical argument s of the Dirichlet series:

$$\begin{aligned} \mathcal{O}_p &\longrightarrow X_S \\ s &\longmapsto \chi^{(s)}, \end{aligned}$$

where $\chi^{(s)}$ is given by $\chi^{(s)}((1+p^\nu)^\alpha) = (1+p^\nu)^{\alpha s} = \exp(\alpha s \log(1+p^\nu))$. The character $\chi^{(s)}$ is defined only for such s for which the series exp is p-adically convergent (i.e. for $|s|_p < p^{\nu-1/(p-1)}$). In this domain of values of the argument we have that $t = (1+p^\nu)^s - 1$. But, for example, for $(1+t)^{p^n} = 1$ there is certainly no such value of s (because $t \neq 1$), so that the s-coordonate parametrizes a smaller neighborhood of the trivial character than the t-coordinate (which parametrizes all wild characters) (see [Man2], [Man3]).

Recall that an analytic function $F : T \longrightarrow \mathbb{C}_p$ ($T = \{z \in \mathbb{C}_p^\times \quad | \quad |z-1|_p < 1\}$), is defined as the sum of a series of the type $\sum_{i\geq 0} a_i(t-1)^i$ ($a_i \in \mathbb{C}_p$), which is assumed to be absolutely convergent for all $t \in T$. The notion of an analytic function is then obviously extended to the whole group X_S by shifts. The function

$$F(t) = \sum_{i=0}^{\infty} a_i(t-1)^i$$

is bounded on T iff all its coefficients a_i are universally bounded. This last fact can be easily deduced for example from the basic properties of the Newton polygon of the series $F(t)$ (see [Kob1], [Vi1], [Vi2]). If we apply to these series the Weierstrass preparation theorem (see [Kob2], [La1], [Man1] and theorem 1.3) we see that in this case the function F has only a finite number of zeroes on T (if it is not identically zero). In particular, consider the torsion subgroup $X_S^{\rm tors} \subset X_S$. This subgroup is discrete in X_S and its elements $\chi \in X_S^{\rm tors}$ can be obviously identified with primitive Dirichlet characters $\chi \bmod M$ such that the support $S(\chi) = S(M)$ of the conductor of χ is contained in S. This identification is provided by a fixed embedding denoted

$$i_p : \overline{\mathbb{Q}}^\times \hookrightarrow \mathbb{C}_p^\times$$

if we note that each character $\chi \in X_S^{\rm tors}$ can be factored through some finite factor group $(\mathbb{Z}/M\mathbb{Z})^\times$:

$$\chi : \mathbb{Z}_S^\times \to (\mathbb{Z}/M\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times \overset{i_p}{\hookrightarrow} \mathbb{C}_p^\times,$$

and the smallest number M with the above condition is the conductor of $\chi \in X_S^{\rm tors}$.

The symbol x_p will denote the composition of the natural projection $\mathbb{Z}_S^\times \to \mathbb{Z}_p^\times$ and of the natural embedding $\mathbb{Z}_p^\times \to \mathbb{C}_p^\times$, so that $x_p \in X_S$ and all integers k can be considered as the characters $x_p^k : y \longmapsto y^k$.

Let us consider a bounded $\mathbb{C}_p$-analytic function F on X_S. The above statement about zeroes of bounded $\mathbb{C}_p$-analytic functions implies now that the function F is uniquely determined by its values $F(\chi_0\chi)$, where χ_0 is a fixed character and χ runs through all elements $\chi \in X_S^{\rm tors}$ with possible exclusion of a finite number of characters in each analyticity component of the decomposition (1.50). This condition is satisfied, for example, by the set of characters $\chi \in X_S^{\rm tors}$ with the S-complete conductor (i.e. such that $S(\chi) = S$), and even for a smaller set of characters, for example for the set obtained by imposing the additional assumption that the character χ^2 is not trivial (see [Man2], [Man3], [Vi1]).

1.4.3 The non-Archimedean Mellin transform

Let μ be a (bounded) $\mathbb{C}_p$-valued measure on $\mathbb{Z}_S^\times$. Then the *non-Archimedean Mellin transform* of the measure μ is defined by

$$L_\mu(x) = \mu(x) = \int_{\mathbb{Z}_S^\times} x \mathrm{d}\mu, \qquad (x \in X_S), \tag{1.51}$$

which represents a bounded $\mathbb{C}_p$-analytic function

$$L_\mu : X_S \longrightarrow \mathbb{C}_p. \tag{1.52}$$

Indeed, the boundedness of the function L_μ is obvious since all characters $x \in X_S$ take values in $\mathcal{O}_p$ and μ also is bounded. The analyticity of this function expresses a general property of the integral (1.51), namely that it depends analytically on the parameter $x \in X_S$. However, we give below a pure algebraic proof of this fact which is based on a description of the Iwasawa algebra. This description will also imply that every bounded $\mathbb{C}_p$-analytic function on X_S is the Mellin transform of a certain measure μ.

1.4.4 The Iwasawa algebra

(see [La1]). Let $\mathcal{O}$ be a closed subring in $\mathcal{O}_p = \{z \in \mathbb{C}_p \quad | \quad |z|_p \leq 1\}$,

$$G = \varprojlim_i G_i, \qquad (i \in I),$$

a profinite group. Then the canonical homomorphism $G_i \xleftarrow{\pi_{ij}} G_j$ induces a homomorphism of the corresponding group rings

$$\mathcal{O}[G_i] \longleftarrow \mathcal{O}[G_j].$$

Then the *completed group ring* $\mathcal{O}[[G]]$ is defined as the projective limit

$$\mathcal{O}[[G]] = \varprojlim_i \mathcal{O}[[G_i]], \qquad (i \in I).$$

Let us consider also the set $\mathrm{Dist}(G, \mathcal{O})$ of all $\mathcal{O}$-valued distributions on G which itself is an $\mathcal{O}$-module and a ring with respect to multiplication given by the *convolution of distributions*, which is defined in terms of families of functions

$$\mu_1^{(i)}, \mu_2^{(i)} : G_i \longrightarrow \mathcal{O},$$

(see the previous section) as follows:

$$(\mu_1 \star \mu_2)^{(i)}(y) = \sum_{y=y_1y_2} \mu_1^{(i)}(y_1)\mu_2^{(i)}(y_2), \qquad (y_1, y_2 \in G_i) \tag{1.53}$$

Recall also that the $\mathcal{O}$-valued distributions are identified with $\mathcal{O}$-valued measures. Now we describe an isomorphism of $\mathcal{O}$-algebras $\mathcal{O}[[G]]$ and $\mathrm{Dist}(G, \mathcal{O})$. In this case when $G = \mathbb{Z}_p$ the algebra $\mathcal{O}[[G]]$ is called the *Iwasawa algebra*.

Theorem 1.10. *(a) Under the same notation as above there is the canonical isomorphism of $\mathcal{O}$-algebras*

$$\mathrm{Dist}(G, \mathcal{O}) \xrightarrow{\sim} \mathcal{O}[[G]]. \tag{1.54}$$

(b) If $G = \mathbb{Z}_p$ then there is an isomorphism

$$\mathcal{O}[[G]] \xrightarrow{\sim} \mathcal{O}[[X]], \tag{1.55}$$

where $\mathcal{O}[[X]]$ is the ring of formal power series in X over $\mathcal{O}$. The isomorphism (1.55) depends on a choice of the topological generator of the group $G = \mathbb{Z}_p$.

1.4.5 Formulas for coefficients of power series

We noticed above that the theorem 1.10 would imply a description of $\mathbb{C}_p$-analytic bounded functions on X_S in terms of measures. Indeed, these functions are defined on analyticity components of the decomposition (1.50) as certain power series with p-adically bounded coefficients, that is, power series, whose coefficients belong to $\mathcal{O}_p$ after multiplication by some constant from $\mathbb{C}_p^\times$. Formulas for coefficients of these series can be also deduced from the proof of the theorem. However, we give a more direct computation of these coefficients in terms of the corresponding measures. Let us consider the component aU_p of the set $\mathbb{Z}_S^\times$ where

$$a \in (\mathbb{Z}/p^\nu\mathbb{Z})^\times \times \prod_{q \neq} \mathbb{Z}_q^\times,$$

and let $\mu_a(x) = \mu(ax)$ be the corresponding measure on U_p defined by restriction of μ to the subset $aU_p \subset \mathbb{Z}_S^\times$. Consider the isomorphism $U_p \cong \mathbb{Z}_p$ given by:

$$y = \gamma^x \quad (x \in \mathbb{Z}_p, y \in U_p),$$

with some choice of the generator γ of U_p (for example, we can take $\gamma = 1+p^\nu$). Let μ'_a be the corresponding measure on $\mathbb{Z}_p$. Then this measure is uniquely determined by values of the integrals

$$\int_{\mathbb{Z}_p} \binom{x}{i} d\mu'_a(x) = a_i, \tag{1.56}$$

with the interpolation polynomials $\binom{x}{i}$, since the $\mathbb{C}_p$-span of the family

$$\left\{\binom{x}{i}\right\} \qquad (i \in \mathbb{Z}, i \geq 0)$$

is dense in $\mathcal{C}(\mathbb{Z}_p, \mathcal{O}_p)$ according to the Mahler's interpolation theorem for continuous functions on $\mathbb{Z}_p$ (see 1.2.1 and [Mah]). Indeed, from the basic properties of the interpolation polynomials it follows that

$$\sum_i b_i \binom{x}{i} \equiv 0 \pmod{p^n} \quad (\text{for all } x \in \mathbb{Z}_p) \Longrightarrow b_i \equiv 0 \pmod{p^n}.$$

We can now apply the abstract Kummer congruences (see proposition 1.7), which imply that for arbitrary choice of numbers $a_i \in \mathcal{O}_p$ there exists a measure with the property (1.56).

On the other hand we state that the Mellin transform L_{μ_a} of the measure μ_a is given by the power series $F_a(t)$ with coefficients as in (1.56), that is

$$\int_{U_p} \chi_{(t)}(y) \mathrm{d}\mu(ay) = \sum_{i=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{i} \mathrm{d}\mu'_a(x) \right) (t-1)^i \tag{1.57}$$

for all wild characters of the form $\chi_{(t)}$, $\chi_{(t)}(\gamma) = t$, $|t-1|_p < 1$. It suffices to show that (1.57) is valid for all characters of the type $y \longmapsto y^m$, where m is a positive integer. In order to do this we use the binomial expansion

$$\gamma^{mx} = (1 + (\gamma^m - 1))^x = \sum_{i=0}^{\infty} \binom{x}{i} (\gamma^m - 1)^i,$$

which implies that

$$\int_{\mathcal{U}_p} y^m \mathrm{d}\mu(ay) = \int_{\mathbb{Z}_p} \gamma^{mx} \mathrm{d}\mu'_a(x) = \sum_{i=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{i} \mathrm{d}\mu'_a(x) \right) (\gamma^m - 1)^i,$$

establishing (1.57).

1.4.6 Example. The S-adic Mazur measure and the non-Archimedean Kubota-Leopoldt zeta function

(see [La1], [Ku-Le], [Le2], [Wa]). Let us first consider a positive integer $c \in \mathbb{Z}_S^\times \cap \mathbb{Z}$, $c > 1$ coprime to all primes in S. Then for each complex number $s \in \mathbb{C}$ there exists a complex distribution μ_s^c on $G_S = \mathbb{Z}_S^\times$ which is uniquely determined by the following condition

$$\mu_s^c(\chi) = (1 - \chi^{-1}(c) c^{-1-s}) L_{M_0}(-s, \chi), \tag{1.58}$$

where $M_0 = \prod_{q \in S} q$ (see 1.3.1). Moreover, the right hand side of (1.58) is holomorphic for all $s \in \mathbb{C}$ including $s = -1$. If s is an integer and $s \geq 0$ then according to criterion of proposition 1.7 the right hand side of (1.58) belongs to the field

$$\mathbb{Q}(\chi) \subset \mathbb{Q}^{\mathrm{ab}} \subset \overline{\mathbb{Q}}$$

generated by values of the character χ, and we get a distribution with values in $\mathbb{Q}^{\mathrm{ab}}$. If we now apply to (1.58) the fixed embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ we get a $\mathbb{C}_p$-valued distribution $\mu^{(c)} = i_p(\mu_0^c)$ which turns out to be an $\mathcal{O}_p$-measure in view of proposition 1.7, and the following equality holds

$$\mu^{(c)}(\chi x_p^r) = i_p(\mu_r^c(\chi)).$$

This identity relates the special values of the Dirichlet L-functions at different non-positive points. The function

$$L(x) = \left(1 - c^{-1} x(c)^{-1}\right)^{-1} L_{\mu^{(c)}}(x) \quad (x \in X_S) \tag{1.59}$$

is well defined and it is holomorphic on X_S with the exception of a simple pole at the point $x = x_p \in X_S$. This function is called the *non-Archimedean zeta-function of Kubota-Leopoldt.* The corresponding measure $\mu^{(c)}$ will be called the *S-adic Mazur measure.*

1.4.7 Measures associated with Dirichlet characters

Let ω mod M be a fixed primitive Dirichlet character such that $(M, M_0) = 1$ with $M_0 = \prod_{q\in S} q$. This section gives a construction of the direct image of the Mazur measure under the natural map $\mathbb{Z}_{\bar S}^\times \to \mathbb{Z}_S^\times$ where $\bar S = S \cup S(M)$, $\bar M_0 = \prod_{q\in \bar S} q$. This construction is used in Chapter III. We show that for any positive integer c with $(c, \bar M_0) = 1$, $c > 1$, there exist $\mathbb{C}_p$-measures $\mu^+(c,\omega)$, $\mu^-(c,\omega)$ on $\mathbb{Z}_S^\times$ which are determined by the following conditions, for $s \in \mathbb{Z}$, $s > 0$:

$$i_p^{-1}\left(\int_{\mathbb{Z}_S^\times} \chi x_p^s \mathrm{d}\mu^+(c,\omega)\right) = (1 - \bar\chi\omega(c)c^{-s})\frac{C_{\omega\bar\chi}}{G(\omega\bar\chi)} \cdot \tag{1.60}$$
$$\cdot \prod_{q\in S\setminus S(\chi)} \left(\frac{1-\chi\bar\omega(q)q^{s-1}}{1-\bar\chi\omega(q)q^{-s}}\right) L^+_{M_0}(s, \bar\chi\omega),$$

and for $s \in \mathbb{Z}$, $s \le 0$,

$$i_p^{-1}\left(\int_{\mathbb{Z}_S^\times} \chi x_p^s \mathrm{d}\mu^-(c,\omega)\right) = (1 - \chi\bar\omega(c)c^{s-1})L^-_{M_0}(s, \bar\chi\omega), \tag{1.61}$$

where

$$L^+_{M_0}(s, \bar\chi\omega) = L_{\bar M}(s, \bar\chi\omega) 2 i^\delta \frac{\Gamma(s)\cos(\pi(s-\delta)/2)}{(2\pi)^s} \tag{1.62}$$
$$L^-_{M_0}(s, \bar\chi\omega) = L_{\bar M}(s, \bar\chi\omega) \tag{1.63}$$

are the normalized Dirichlet L-functions with $\delta \in \{0,1\}$, and $\bar\chi\omega(-1) = (-1)^\delta$. The function $G(\omega\bar\chi)$ denote the Gauss sum of the Dirichlet character $\omega\bar\chi$. The functions (1.60), (1.61) satisfy the functional equation

$$L^-_{M_0}(1-s, \bar\chi\omega) = \prod_{q\in S\setminus S(\chi)} \left(\frac{1-\chi\bar\omega(q)q^{s-1}}{1-\bar\chi\omega(q)q^{-s}}\right) L^+_{M_0}(s, \bar\chi\omega).$$

Indeed, by the definition of the $\bar S$-adic Mazur measure μ^c on $\mathbb{Z}_S^\times$, (1.60) and (1.61) are given by

$$\int_{\mathbb{Z}_S^\times} \mathrm{d}\mu^+(c,\omega) \overset{\mathrm{def}}{=} \int_{\mathbb{Z}_{\bar S}^\times} x x_p^{-1}\omega^{-1}\mathrm{d}\mu^c,$$
$$\int_{\mathbb{Z}_S^\times} x\mathrm{d}\mu^-(c,\omega) \overset{\mathrm{def}}{=} \int_{\mathbb{Z}_{\bar S}^\times} x^{-1}\omega\mathrm{d}\mu^c,$$

where $x \in X_S$ and $X_{\bar S}$ is viewed as a subgroup of X_S.

1.5 Admissible measures and their Mellin transform

1.5.1 Non-Archimedean integration

This construction was nicely explained by J. Coates and B. Perrin-Riou [Co-PeRi] using the Fourier transform of distributions. Let S be a finite set of primes containing p. The set on which our non-Archimedean zeta functions are defined is the $\mathbb{C}_p$-adic analytic Lie group

$$X_S = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}_S, \mathbb{C}_p^\times), \tag{1.64}$$

where Gal_S is the Galois group of the maximal abelian extension of $\mathbb{Q}$ unramified outside S and infinity. Now we recall the notion of h-admissible measures on Gal_S and properties of their Mellin transform. These Mellin transforms are certain p-adic analytic functions on the $\mathbb{C}_p$-analytic group X_S. Recall that by class field theory the group Gal_S is described as the projective limit

$$\mathrm{Gal}_S \cong \varprojlim_M (\mathbb{Z}/M\mathbb{Z})^\times = \mathbb{Z}_S^\times, \tag{1.65}$$

where M runs over integers with support in the set of primes S (i.e. $S(M) \subset S$). The canonical $\mathbb{C}_p$-analytic structure on X_S is obtained by shifts from the obvious $\mathbb{C}_p$-analytic structure on the subgroup $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times) \subset X_S$. We regard the elements of finite order $\chi \in X_S^{\mathrm{tors}}$ as Dirichlet characters whose conductor $c(\chi)$ may contain only primes in S, by means of the decomposition

$$\chi : \mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times \longrightarrow \mathbb{Z}_S^\times \longrightarrow \overline{\mathbb{Q}}^\times \xrightarrow{i_\infty} \mathbb{C}^\times, \tag{1.66}$$

where i_∞ is a fixed embedding. The character $\chi \in X_S^{\mathrm{tors}}$ form a discrete subgroup $X_S^{\mathrm{tors}} \subset X_S$. We shall need also the natural homomorphism

$$x_p : \mathbb{Z}_S^\times \longrightarrow \mathbb{Z}_p^\times \longrightarrow \mathbb{C}_p^\times, \quad x_p \in X_S, \tag{1.67}$$

so that all integers $k \in \mathbb{Z}$ can be regarded as characters of the type $x_p^k : y \longmapsto y^k$.

Recall that a p-adic measure on $\mathbb{Z}_S^\times$ may be regarded as a bounded $\mathbb{C}_p$-linear form μ on the space $\mathcal{C}(\mathbb{Z}_S^\times, \mathbb{C}_p)$ of all continuous $\mathbb{C}_p$-valued functions

$$\begin{aligned} \mathcal{C}(\mathbb{Z}_S^\times, \mathbb{C}_p) &\longrightarrow \mathbb{C}_p \\ \varphi &\longmapsto \mu(\varphi) = \int_{\mathbb{Z}_S^\times} \varphi \mathrm{d}\mu \end{aligned}$$

which is uniquely determined by its restriction to the subspace $\mathrm{Step}(\mathbb{Z}_S^\times, \mathbb{C}_p)$ of locally constant functions. We denote by $\mu(a + (M))$ the value of μ on the characteristic function of the set

$$a + (M) = \{x \in \mathbb{Z}_S^\times \quad | \quad x \equiv a \pmod M\} \subset \mathbb{Z}_S^\times.$$

The Mellin transform L_μ of μ is a bounded analytic function

$$\begin{aligned} L_\mu : X_S &\longrightarrow \mathbb{C}_p \\ \chi &\longmapsto L_\mu(\chi) = \int_{\mathbb{Z}_S^\times} \chi \mathrm{d}\mu \end{aligned}$$

on X_S, which is uniquely determined by its values $L_\mu(\chi)$ for the characters $\chi \in X_S^{\mathrm{tors}}$.

1.5.2 h-admissible measure

A more delicate notion of an *h-admissible measure* was introduced by Y. Amice, J. Vélu and M.M. Višik (see [Am-Ve], [Vi1]). Let $\mathcal{C}(\mathbb{Z}_S^\times, \mathbb{C}_p)$ denote the space of $\mathbb{C}_p$-valued functions that can be locally represented by polynomials of degree less than a natural number h of the variable $x_p \in X_S$ introduced above.

Definition 1.11. *A $\mathbb{C}_p$-linear form $\mu : \mathcal{C}^h(\mathbb{Z}_S^\times, \mathbb{C}_p) \longrightarrow \mathbb{C}_p$ is called an h-admissible measure if for all $r = 0, 1, \cdots, h-1$ the following growth condition is satisfied*

$$\left|\sup_{a\in\mathbb{Z}_S^\times} \int_{a+(M)} (x_p - a_p)^r \mathrm{d}\mu\right|_p = o\left(|M|_p^{r-h}\right).$$

Note that the notion of a bounded measure is covered by the case $h = 1$, but the set of 1-admissible measures is bigger; it consists of the so called measures of bounded growth (see [Man2], [Vi1]), which are characterized by the property that they grow on open compact sets $a + (M) \subset \mathbb{Z}_S^\times$ slower than $\mathrm{o}(|M|_p^{-1})$. We know (essentially due to Y. Amice, J. Vélu and M.M. Višik) that each h-admissible measure can be uniquely extended to a linear form on the $\mathbb{C}_p$-space of all locally analytic functions so that one can associate to its Mellin transform

$$\begin{aligned} L_\mu : X_S &\longrightarrow \mathbb{C}_p \\ \chi &\longmapsto L_\mu(\chi) = \int_{\mathbb{Z}_S^\times} \chi \mathrm{d}\mu \end{aligned}$$

which is a $\mathbb{C}_p$-analytic function on X_S of the type $\mathrm{o}(\log(x_p^h))$. Moreover, the measure μ is uniquely determined by the special values of the type $L_\mu(\chi x_p^r)$ with $\chi \in X_S^{\mathrm{tors}}$ and $r = 0, 1, \cdots, h-1$. First example of *h-admissible measures* and their L-functions concerned the L-function

$$L_f(s, \chi) = \sum_{n=1}^{\infty} \chi(n) a_n n^{-s}$$

of an elliptic normalized Hecke cusp eigenform

$$f(z) = \sum_{n=1}^{\infty} a_n \exp(2i\pi z) \qquad (\mathrm{Im}(z) > 0)$$

with a_p divisible by p (the *supersingular case*). This notion is used in Chapter 4.

We can consider another example, provided by the analytic function $\log(x)$. For any$s \in \mathbb{N}$ let us define a distribution μ_s by

$$\mu_s(\chi) = \mu(\chi x_p^s) = s\log(1 + p^\nu), \qquad \text{for all } \chi \in X_S^{\mathrm{tors}} \quad \text{and } s \in \mathbb{N}.$$

This sequence of distributions turns out to be a 2-admissible measure. Indeed, for $r = 0, 1$

$$\int_{a+(M)} (x_p - a_p)^r \mathrm{d}\mu = \sum_{\chi \bmod M} \chi^{-1}(a) \int_{\mathbb{Z}_S^\times} \chi(x)(x_p - a_p)^r \mathrm{d}\mu$$
$$= \begin{cases} 0, & \text{if } r = 0, \\ \log(1+p^\nu) \sum_{\chi \bmod M} \chi^{-1}(a), & \text{if } r = 1 \end{cases}$$

But, in the case $r = 1$ the last sum $\sum_{\chi \bmod M} \chi^{-1}(a)$ is equal to 0 if $a \neq 1$, and if $a = 1$ then this sum is equal to $\varphi(M)$. So we see that the conditions of the definition 1.11 are satisfied, so that we obtain a 2-admissible measure. If we look now at its Mellin transform

$$L_\mu(x) = \int_{\mathbb{Z}_S^\times} x \mathrm{d}\mu,$$

we see immediately that $L_\mu(\chi x_p) = \log(x_p)$ which is a $\mathbb{C}_p$-analytic function on $X_S^\times$ of the type $\mathrm{o}(\log(x_p)^2)$.

1.6 Complex valued distributions, associated with Euler products

In this section we give a general construction of distributions, attached to rather arbitrary Euler products. This construction provides a generalization of measures, which were first introduced by Y.I. Manin, B. Mazur and H.P.F. Swinnerton-Dyer (see [Man2], [Maz-SD]). Our construction ([Pa2], [Pa3]) was already successfully used in several problems concerning the p-adic analytic interpolation of Dirichlet series (see [Ar], [Co-Schm], [Sc]).

1.6.1 Dirichlet series

Let S be a fixed finite set of primes and

$$\mathcal{D}(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad (s, a_n \in \mathbb{C}) \tag{1.68}$$

be a Dirichlet series with the following multiplicativity property of its coefficients a_n:

$$\mathcal{D}(s) = \prod_{q \in S} F_q(q^{-s})^{-1} \sum_{\substack{n=1 \\ (n,S)=1}} a_n n^{-s}, \tag{1.69}$$

where the condition $(n, S) = 1$ means that n is not divisible by any prime in S, and $F_q(X)$ are polynomials with the constant term equal to 1:

$$F_q(X) = 1 + \sum_{i=1}^{m_q} A_{q,i} X^i. \tag{1.70}$$

We assume also that the series (1.68) is absolutely convergent in some right half plane $\mathrm{Re}(s) > 1 + c$ ($c \in \mathbb{R}$). This assumption is satisfied in most cases, for example, when the coefficients a_n satisfy the estimate $|a_n| = \mathcal{O}(n^c)$. For a Dirichlet character $\chi : (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}^\times$ modulo $M \geq 1$ the twisted Dirichlet series is defined by

$$\mathcal{D}(s, \chi) = \sum_{n=1}^{\infty} \chi(n) a_n n^{-s}. \tag{1.71}$$

For all $s \in \mathbb{C}$ such that the series (1.68) is absolutely convergent, let us define the function $P_s : \mathbb{Q} \to \mathbb{C}$ by the equality

$$P_s(x) = \sum_{n=1}^{\infty} e(nx) a_n n^{-s} \quad (e(x) = \exp(2i\pi x)). \tag{1.72}$$

Using the functions (1.72) we construct $\mathbb{C}$-valued distributions μ_s on the compact group $\mathbb{Z}_S^\times$ such that for every primitive Dirichlet character χ viewed as a homomorphism $\chi : \mathbb{Z}_S^\times \to \mathbb{C}^\times$ the value of the series $\mathcal{D}(s, \bar{\chi})$ at s with $\mathrm{Re}(s) > 1 + c$ is canonically expressed in terms of the integral

$$\int_{\mathbb{Z}_S^\times} \chi \mathrm{d}\mu_s \stackrel{\mathrm{def}}{=} \sum_{\substack{a \bmod M \\ (a,S)=1}} \chi(a) \mu_s(a + (M)).$$

Let for every $q \in S$, $\alpha(q)$ denote a fixed root of the inverse polynomial

$$X^{m_q} F_q(X^{-1}) = X^{m_q} + \sum_{i=1}^{m_q} A_{q,i} X^{m_q - i}$$

(that is, an inverse root of $F_q(X)$). We suppose that $\alpha(q) \neq 0$ for every $q \in S$, and let us extend the definition of numbers $\alpha(n)$ to all positive integers whose support is contained in S (by multiplicativity):

$$\alpha(n) = \prod_{q \in S} \alpha(q)^{\mathrm{ord}_q(n)} \quad (S(q) \subset S).$$

Let us define an auxiliary polynomial

$$H_q(X) = 1 + \sum_{i=1}^{m_q - 1} B_{q,i} X^i \tag{1.73}$$

by means of the relation

$$F_q(X) = (1 - \alpha(q) X) H_q(X), \tag{1.74}$$

which implies the identities

$$B_{q,i} = -\sum_{j=I+1}^{m_q} A_{q,j}\alpha(q)^{i-j} \quad (i = 1, \cdots, m_q - 1) \tag{1.75}$$

for the coefficients of the polynomial (1.73). Let us also introduced the following finite Euler product

$$\sum_{S(n)\subset S} B^S(n)n^{-s} = \prod_{q\in S} H_q(q^{-s}), \tag{1.76}$$

in which the coefficients $B^S(n)$ are given by means of (1.75), namely,

$$B^S(n) = \prod_{q\in S} B^S(q^{\mathrm{ord}_q(n)}) \quad (S(n) \subset S),$$

with

$$B^S(q^i) = \begin{cases} B_{q,i}, & \text{for } i < m_q \\ 0, & \text{otherwise.} \end{cases} \tag{1.77}$$

Now we state the main result of the section.

Theorem 1.12. *(a) For any choice of the inverse roots $\alpha(q) \neq 0$ $(q \in S)$, and for any s from the convergency region of the series (1.68) there exists a distribution $\mu_s = \mu_{s,\alpha}$ on $\mathbb{Z}_S^\times$ whose values on open compact subsets of the type $a + (M) \subset \mathbb{Z}_S^\times$ are given by the following*

$$\mu_s(a + (M)) = \frac{M^{s-1}}{\alpha(M)} \sum_{S(n)\subset S} B^S(n)P_s\left(\frac{a_n}{M}\right) n^{-s}, \tag{1.78}$$

so that the sum in (1.78) is finite and the numbers $B^S(n)$ are defined by (1.77).

(b) For any primitive Dirichlet character χ viewed as a function $\chi : \mathbb{Z}_S^\times \to \mathbb{C}^\times$ the following equality holds

$$\int_{\mathbb{Z}_S^\times} \chi d\mu_s = \prod_{q\in S\setminus S(\chi)} \left(1 - \chi(q)\alpha(q)^{-1}q^{-s}\right) H_q\left(\bar{\chi}(q)q^{-s}\right) \frac{C_\chi^{s-1}}{\alpha(C_\chi)} G(\chi)\mathcal{D}(s, \bar{\chi}), \tag{1.79}$$

with

$$G(\chi) = \sum_{a \bmod C_\chi} \chi(a)e\left(\frac{a}{C_\chi}\right)$$

being the Gauss sum, C_χ the conductor, and $S(\chi)$ the support of the conductor of χ.

Remark 1.13. The distribution μ_s can be obtained as the Fourier transform of the standard zeta-distribution attached to the Diriclet series

$$\mathcal{D}(s, \chi) = \prod_{q\in S} H_q(\chi(q)q^{-s}) = \sum_{n\geq 1} \chi(n)a_0(n)n^{-s},$$

where $a_0(qn) = \alpha(q)a_0(n)$ for each $n \in \mathbb{N}$ and $q \in S$ (see [Co-PeRi]).

Proof of theorem 1.12

The following proof of this theorem differs from that given in [Pa2] and is based on the compatibility criterion (proposition 1.7). We check that the sum

$$\sum_{\substack{a \bmod M \\ (a,S)=1}} \chi(a)\mu_s(a+(M)), \tag{1.80}$$

does not depend on the choice of a positive integer M with the condition $C_\chi|M$, $S(M) = S$. This will be provided by a calculation which also implies that (1.80) coincides with the right hand side of (1.79) (and therefore is independent of M).

Lemma 1.14. *For an arbitrary positive integer n and $C_\chi|M$ put*

$$G_{n,M} = \sum_{\substack{a \bmod M \\ (a,S)=1}} \chi(a)e(an/M).$$

Then the following holds

$$G_{n,M}(\chi) = \frac{M}{C_\chi}G(\chi) \sum_{d|(M/C_\chi)} \mu(d)d^{-1}\chi(d)\delta\left(\frac{dn}{(M/C_\chi)}\right)\bar{\chi}\left(\frac{dn}{(M/C_\chi)}\right),$$

in which μ denotes the Möbius function, $\delta(x) = 1$ or 0 according as $x \in \mathbb{Z}$ or not, and we assume that the character χ is primitive modulo C_χ.

Proof. The proof of this lemma is deduced from the well known property of the Möbius function:

$$\sum_{d|n} \mu(d) = \begin{cases} 1, \text{ if } n = 1; \\ 0, \text{ if } n > 1. \end{cases}$$

Consequently, $G_{n,M}$ takes the following form:

$$\sum_{\substack{d|(a,M) \\ a \bmod M}} \mu(d)\chi(a)e(an/M) =$$

$$\begin{aligned}
&= \sum_{d|M} \mu(d) \sum_{a_1 \bmod M/d} \chi(da_1)e(da_1n/M) \\
&= \sum_{d|M} \mu(d)d^{-1}\chi(d) \sum_{a_1 \bmod M} \chi(a_1)e(da_1n/M) \\
&= \sum_{d|(M/C_\chi)} \mu(d)d^{-1}\chi(d)\delta\left(\frac{dn}{(M/C_\chi)}\right) \sum_{a_1 \bmod M} \chi(a_1)e(da_1n/M),
\end{aligned}$$

since $\chi(a_1)$ depend only on $a_1 \bmod C_\chi$, and

$$e(a_1 dn/M) = e((a_1/C_\chi)(dn/(M/C_\chi))).$$

In the above equality we changed the order of summation, then we replaced the index of summation a by da_1 and extended a system of residue classes $a_1 \bmod M/d$ to a system $a_1 \bmod M$. Now we transform the summation into that one modulo C_χ. It remains to use the well known property of Gauss sums (see, for example, [Sh1], lemma 3.63):

$$G_{n,C_\chi}(\chi) = \bar{\chi}(n)G(\chi),$$

establishing the lemma.

In order to deduce the theorem, we now transform (1.80), taking into account the definition (1.78) and lemma 1.14:

$$\frac{M^{s-1}}{\alpha(M)} \sum_{\substack{a \bmod M \\ (a,M)=1}} \chi(a) \sum_n B^S(n)n^{-s} \sum_{n_1} a_{n_1} e\left(\frac{ann_1}{M}\right) n_1^{-s} = \tag{1.81}$$

$$= \frac{M^{s-1}}{\alpha(M)} \sum_n \sum_{n_1} B^S(n)n^{-s} a_{n_1} n_1^{-s} G_{nn_1,M}(\chi)$$
$$= \frac{M^s}{\alpha(M)C_\chi} G(\chi) \sum_n \sum_{n_1} B^S(n)n^{-s} a_{n_1} n_1^{-s} \cdot$$
$$\cdot \sum_{d|(M/C_\chi)} \frac{\mu(d)}{d} \chi(d) \delta\left(\frac{dnn_1}{(M/C_\chi)}\right) \bar{\chi}\left(\frac{dnn_1}{(M/C_\chi)}\right).$$

From the last formula we see that non-vanishing terms in the sum over n and n_1 must satisfy the condition $(M/C_\chi d)|nn_1$. Let us now split n_1 into two factors $n_1 = n_1' \cdot n_1''$ so that $S(n_1') \subset S$, and $(n_1'', S) = 1$. Then

$$\bar{\chi}\left(\frac{dnn_1}{(M/C_\chi)}\right) = \bar{\chi}(n_1'')\bar{\chi}\left(\frac{nn_1'}{(M/C_\chi d)}\right) \tag{1.82}$$

and

$$\delta\left(\frac{dnn_1}{(M/C_\chi)}\right) = \delta\left(\frac{nn_1'}{(M/C_\chi d)}\right), \tag{1.83}$$

since $(n_1'', M) = 1$. According to (1.69) one has

$$\sum_{S(n_1') \subset S} a_{n_1'} {n_1'}^{-s} = \prod_{q \in S} F_q(q^{-s})^{-1}. \tag{1.84}$$

Now we use the definition of the finite Euler product (1.76) and of the polynomials $H_q(X)$ which we rewrite here in the form

$$\sum_n B^S(n)n^{-s} \prod_{q \in S} F_q(q^{-s})^{-1} = \prod_{q \in S} (1 - \alpha(q)q^{-s})^{-1}.$$

Consequently,

$$\sum_n B^S(n)n^{-s} \sum_{S(n_1')\subset S} a_{n_1'} {n_1'}^{-s} = \sum_{S(n_2)\subset S} a_{n_2} n_2^{-s}$$

and for $S(n_2) \subset S$ we have that

$$\alpha(n_2) = \sum_{n_2 = nn_1'} B^S(n) a_{n_1'}. \tag{1.85}$$

Keeping in mind (1.82) and (1.83) we transform (1.81) to the following

$$\frac{M^s}{\alpha(M)C_\chi} G(\chi) \sum_n \sum_{d|(M/C_\chi)} \frac{\mu(d)}{d}\chi(d) \sum_{(n_1'',S)=1} \bar\chi(n_1'') a_{n_1''} {n_1''}^{-s} \cdot \tag{1.86}$$
$$\cdot \sum_{n,n_1'} B^S(n) a_{n_1'} (nn_1')^{-s} \delta\left(\frac{nn_1'}{(M/C_\chi d)}\right) \bar\chi\left(\frac{nn_1'}{(M/C_\chi d)}\right).$$

Now we transform (1.86) with the help of the relation (1.85), taking into account that non-zero summands can only occur for such $n_2 = nn_1'$ which are divisible by $M/(C_\chi d)$, (i.e. we put $n_2 = (M/C_\chi d)n_3$, $S(n_3) \subset S$). We also note that by the definition of our Dirichlet series we have

$$\sum_{n,n_1'} B^S(n) a_{n_1'} (nn_1')^{-s} = \mathcal{D}(s,\bar\chi) \prod_{q\in S\setminus S(\chi)} F_q(\bar\chi(q)q^{-s}).$$

Therefore (1.86) transforms to the following

$$\frac{M^s}{\alpha(M)C_\chi} G(\chi) \prod_{q\in S\setminus S(\chi)} F_q(\bar\chi(q)q^{-s}) \sum_{d|(M/C_\chi)} \frac{\mu(d)}{d}\chi(d)\cdot$$
$$\cdot \sum_{S(n_3)\subset S} \bar\chi(n_3)\alpha\left(\frac{n_3 M}{C_\chi d}\right)\left(\frac{n_3 M}{c_\chi d}\right)^{-s} =$$
$$= \frac{C_\chi^{s-1}}{\alpha(C_\chi)} G(\chi)\mathcal{D}(s,\bar\chi) \prod_{q\in S\setminus(\chi)} F_q(\bar\chi(q)q^{-s}) \sum_{d|(M/C_\chi)} \mu(d)d^{s-1}\chi(d)\alpha(d)^{-1}\cdot$$
$$\cdot \sum_{S(n_3)\subset S} \bar\chi(n_3)\alpha(n_3)n_3^{-s}.$$

The proof of the theorem is accomplished by noting that

$$\begin{aligned}
\sum_{d|(M/C_\chi)} \mu(d)d^{s-1}\chi(d)\alpha(d)^{-1} &= \prod_{q\in S\setminus S(\chi)} (1-\chi(q)\alpha(q)^{-1}q^{s-1}),\\
\sum_{S(n_3)\subset S} \bar\chi(n_3)\alpha(n_3)n_3^{-s} &= \prod_{q\in S\setminus S(\chi)} (1-\bar\chi(q)\alpha(q)q^{-s})^{-1}\\
&= \prod_{q\in S\setminus S(\chi)} F_q(\bar\chi(q)q^{-s})^{-1} H_q(\bar\chi(q)q^{-s}).
\end{aligned}$$

1.6.2 Concluding remarks

This construction admits a generalization [Pa4] to the case of rather general Euler products over prime ideals in algebraic number fields. These Euler products have the form

$$\mathcal{D}(s) = \sum_{\mathfrak{n}} a_{\mathfrak{n}} \mathcal{N}(\mathfrak{n})^{-s} = \prod_{\mathfrak{p}} F_{\mathfrak{p}}(\mathcal{N}(\mathfrak{p})^{-s})^{-1},$$

where $\mathfrak{n}$ runs over the set of integrals ideals, and $\mathfrak{p}$ over the set of prime ideals of integers $\mathcal{O}_K$ of a number field K, with $\mathcal{N}(\mathfrak{n})$ denoting the absolute norm of an ideal $\mathfrak{n}$, and $F_{\mathfrak{p}} \in \mathbb{C}[X]$ being polynomials with the condition $F_{\mathfrak{p}}(0) = 1$. In [Pa4] we constructed certain distributions, which provide integral representations for special values of Dirichlet series of the type

$$\mathcal{D}(s, \chi) = \sum_{\mathfrak{n}} \chi(\mathfrak{n}) a_{\mathfrak{n}} \mathcal{N}(\mathfrak{n})^{-s},$$

where χ denote a Hecke character of finite order, whose conductor consists only of prime ideals belonging to a fixed finite set S of non-Archimedean places of K. The main result of [Pa4] provides a generalization of theorem 4.2 of the earlier work of Yu.I. Manin [Man3].

In the construction of non-Archimedean convolutions of Hilbert modular forms given in chapter 4, we give another approach to local distributions, which differs from the given above and is applicable only to certain Dirichlet series (namely, to convolutions of Rankin type). However, there is an interesting link between these two approaches, which is based on a general construction of p-adic distributions attached to motives. It turns out that both types of distributions can be obtained using the Fourier transform in the distribution space (see [Co-PeRi], [Co-Schn]).